# Random Variate Generation for Exponential and Gamma Tilted Stable Distributions

YAN QU, Peking University
ANGELOS DASSIOS, London School of Economics and Political Science
HONGBIAO ZHAO, Shanghai University of Finance and Economics

We develop a new efficient simulation scheme for sampling two families of tilted stable distributions: exponential tilted stable (ETS) and gamma tilted stable (GTS) distributions. Our scheme is based on two-dimensional single rejection. For the ETS family, its complexity is uniformly bounded over all ranges of parameters. This new algorithm outperforms all existing schemes. In particular, it is more efficient than the well-known double rejection scheme, which is the only algorithm with uniformly bounded complexity that we can find in the current literature. Beside the ETS family, our scheme is also flexible to be further extended for generating the GTS family, which cannot easily be done by extending the double rejection scheme. Our algorithms are straightforward to implement, and numerical experiments and tests are conducted to demonstrate the accuracy and efficiency.

CCS Concepts: • **Mathematics of computing** → *Random number generation*;

Additional Key Words and Phrases: Monte Carlo simulation, random variate generation, two-dimensional single rejection, tempered stable distribution, Lévy process

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## 1 INTRODUCTION

The family of positive stable distributions, which was introduced by Lévy [1925], is an important mathematical tool for capturing heavy tails of observations from reality, such as financial time series of price returns. A series of influential work by Mandelbrot [1961, 1963a, b] had demonstrated its importance for potential applications in finance and economics. However, one crucial problem, as later pointed by many scholars, is its infinite moments, which would be especially problematic for pricing assets such as options. To deal with this issue, the tail of a positive stable distribution

Authors' addresses: Y. Qu (corresponding author), School of Mathematical Sciences, Peking University, Beijing 100871, China; email: y.qu3@lse.ac.uk; A. Dassios, Department of Statistics, London School of Economics and Political Science, Houghton Street, London WC2A 2AE, UK; email: a.dassios@lse.ac.uk; H. Zhao, School of Statistics and Management, Shanghai University of Finance and Economics, No. 777 Guoding Road, Shanghai 200433, China; and Shanghai Institute of International Finance and Economics, No. 777 Guoding Road, Shanghai 200433, China; email: h.zhao1@lse.ac.uk.

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should be tilted (or tempered); see discussions in Carr and Wu [2003] and Wu [2006]. A very popular version of the tilted stable distribution is the so-called **exponentially tilted stable (ETS)** distribution, which was initially proposed by Tweedie [1984] and Hougaard [1986]. It plays a key role in mathematical statistics, as a model for randomness used by Bayesians, and in economic models [Devroye 2009]. Furthermore, the family of ETS distributions has become a fundamental component to be used to construct many useful stochastic processes, which have numerous applications in finance and many other fields. For example, ETS-driven *non-Gaussian Ornstein-Uhlenbeck processes* are used for modelling stochastic volatilities of asset prices and contagion risk processes (see Andrieu et al. [2010]; Barndorff-Nielsen and Shephard [2002, 2003]; Qu et al. [2019, 2021]; Todorov [2015]). More recently, ETS-driven *Lévy subordinators* have been adopted for modeling the stochastic-time clocks in a series of time-changed models proposed by Li and Linetsky [2013, 2014, 2015] and Mendoza-Arriaga and Linetsky [2014, 2016]. In addition, ETS distributions as key members of infinitely divisible distributions are closely connected with characteristic kernels, which play an import role in machine learning applications (see Nishiyama and Fukumizu [2016]).

The simulation design for sampling ETS distributions without bias has recently received attention in the literature. The most widely used and trivial algorithm probably is the **simple stable rejection (SSR)** scheme, which is developed by a simple combination of the well known *Zolotarev's integral representation* [Zolotarev 1966] and an **acceptance-rejection (A/R)** scheme (see Brix [1999]). Hofert [2011a] suggested a **fast rejection (FR)** algorithm to enhance the SSR scheme. However, the complexities[1] of SSR and FR are unbounded, which obviously limits their applicability, as they would become extremely inefficient for some parameter choices. To overcome this problem, Devroye [2009] developed a novel scheme based on **double rejection (DR)** such that the complexity is uniformly bounded. Alternatively, in this article, we design a new scheme for ETS distributions based on **two-dimensional single rejection (SR)**.[2] The complexity of our SR scheme is also uniformly bounded, and remarkably, it outperforms the DR scheme for all ranges of parameters. More precisely, the complexity of our SR scheme is roughly bounded by 4.2154 over all parameters, which is smaller than the one for the DR scheme. Furthermore, we can easily extend our scheme for sampling **gamma tilted stable (GTS)** distributions, which cannot easily be done by extending the DR scheme since the R distribution suggested in Devroye [2009] has been replaced by other distribution for GTS. The GTS distribution was first introduced by Barndorff-Nielsen and Shephard [2001] for modeling stochastic volatility of financial time series. The first simulation algorithm was just developed recently by Favaro et al. [2015], which is based on the decomposition for the GTS. Since our algorithm for the GTS does not depend on the ETS simulation scheme, it is much easier to set up and implement than the one in Favaro et al. [2015].

The article is structured as follows. In Section 2, we provide preliminaries for the positive stable distribution, **exponential tilted stable (ETS)** distribution, introduce the general two-dimensional SR framework, and develop several simulation schemes for sampling ETS distributions. In Section 3, we analyze the performances of several proposed algorithms with regard to different choices of tilting and stability parameters, then, by optimally combining these schemes, we propose a super efficient uniformly bounded scheme to sample ETS variables over the whole range of stability and tilting parameters. In Section 4, we extend the simulation idea

[1]The *complexity* of an algorithm is the expected number of iterations before halting (see Law [2015, Ch. 8]). In particular for the A/R methodology, its complexity is exactly the associated A/R constant.

[2]This idea originates from the approach of distributional decomposition and transformation adopted by Dassios et al. [2018], where they tailored efficient simulation algorithms for some special ETS classes (see also Dassios et al. [2020] for this approach).

from ETS distributions to GTS distributions. In Section 5, extensive numerical experiments for our algorithms, as well as the associated comparisons with other schemes, are carried out and reported in detail. Section 6 draws a brief conclusion for this article.

## 2 PRELIMINARIES

### 2.1 ETS Distributions

A positive stable random variable $S_\alpha$ with the stability index $\alpha \in (0,1)$ has the Laplace transform

$$\mathbb{E}\left[e^{-vS_\alpha}\right] = e^{-v^\alpha}, \qquad v \in \mathbb{R}^+. \tag{1}$$

The density function of $S_\alpha$ has the well-known *Zolotarev's integral representation* [Zolotarev 1986],

$$f_\alpha(s) = \frac{1}{\pi}\int_0^\pi \frac{\alpha}{1-\alpha} B(u)^{\frac{1}{1-\alpha}} s^{-\frac{1}{1-\alpha}} e^{-B(u)^{\frac{1}{1-\alpha}} s^{-\frac{\alpha}{1-\alpha}}} \mathrm{d}u, \qquad s \in \mathbb{R}^+, \tag{2}$$

where $B(u)$ is defined as

$$B(u) := \frac{\sin^\alpha(\alpha u)\sin^{1-\alpha}((1-\alpha)u)}{\sin u}, \qquad u \in [0,\pi].$$

The associated ETS random variable $S_{\alpha,\lambda}$ is defined through the exponentially tilting distribution of $S_\alpha$ with tilting parameter $\lambda \in \mathbb{R}^+$. The Laplace transform of $S_{\alpha,\lambda}$ therefore is

$$\mathbb{E}\left[e^{-vS_{\alpha,\lambda}}\right] = e^{\lambda^\alpha - (\lambda+v)^\alpha}, \tag{3}$$

and the density function of $S_{\alpha,\lambda}$ is given by

$$f_{\alpha,\lambda}(s) = e^{\lambda^\alpha - \lambda s} f_\alpha(s) = \int_0^\pi f(s,u)\mathrm{d}u, \tag{4}$$

where $f(s,u)$ is the bivariate density function of $(S,U)$ in $(s,u)$ on $[0,\infty)\times[0,\pi]$—that is,

$$f(s,u) = \frac{\alpha e^{\lambda^\alpha}}{(1-\alpha)\pi} B(u)^{\frac{1}{1-\alpha}} s^{-\frac{1}{1-\alpha}} \exp\left(-B(u)^{\frac{1}{1-\alpha}} s^{-\frac{\alpha}{1-\alpha}} - \lambda s\right). \tag{5}$$

This $S_{\alpha,\lambda}$ cannot be easily simulated directly due to the Zolotarev's integral representation (2). However, we can use our two-dimensional A/R scheme to sample $(S,U)$ and return $S$ to sample $S_{\alpha,\lambda}$ instead.

*Remark 2.1.* Other works in the literature may use an alternative parameterization for the ETS distribution with Laplace transform

$$\mathbb{E}\left[e^{-vS_{\alpha,\lambda,\theta}}\right] = e^{\theta[\lambda^\alpha - (\lambda+v)^\alpha]},$$

where $\theta \in \mathbb{R}^+$ is a new parameter. Without loss of generality, we set $\theta = 1$ in this work, since

$$S_{\alpha,\lambda,\theta} \overset{\mathcal{D}}{=} \theta^{\frac{1}{\alpha}} S_{\alpha,\lambda\theta^{\frac{1}{\alpha}}}$$

(see Devroye [2009, p. 12]).

### 2.2 Two-Dimensional SR Scheme

Several competing algorithms for simulating ETS distributions have been proposed in the literature (i.e., the SSR scheme [Brix 1999], FR scheme [Hofert 2011a], and DR scheme [Devroye 2009]. These algorithms are unbiased and can produce very accurate samples. However, each of them has its own advantages and limitations. For the SSR scheme, since the expected complexity is exponentially increasing, the algorithm has a very poor acceptance rate for a large value of tilting parameter $\lambda$. For the FR scheme, it works well for a small value of $\alpha$, but its complexity is of order $O(\lambda^{\alpha})$, which is clearly unbounded. For the DR scheme, although the complexity is uniformly bounded, the upper bound is still large. In particular, when $\alpha$ is close to 0, the simulation becomes much less efficient. Comparing with the SSR scheme and the FR scheme, the DR scheme is more difficult for a practitioner to implement, as the procedure is rather complicated. Hence, it is of great interest to develop a simpler and more efficient algorithm with lower uniformly bounded complexity for all $\alpha \in (0, 1)$ and $\lambda \in \mathbb{R}^+$, and this is the aim of our work.

Given the density function of $S_{\alpha,\lambda}$ in (4) with the joint density function $f(s, u)$ of a bivariate variable $(S, U)$ in (5), we can use the two-dimensional A/R scheme to sample $(S, U)$ by choosing an appropriate bivariate envelope $(S', U')$ with density $g(s, u)$. Therefore, we can use the following general simulation framework, Algorithm 2.1, to sample the associated marginal variate $S$.

**ALGORITHM 2.1:** Two-Dimensional SR Framework

(1) **set** $C = \sup_{s,u} \{f(s, u)/g(s, u)\}$
(2) **repeat**{
(3) **sample** $(S, U)$ with density $g(s, u)$, $V \sim \mathcal{U}(0, 1)$
(4) **if** $(V \leq \frac{f(S,U)}{Cg(S,U)})$ **break**
(5) }
(6) **return** $S$

The expected complexity, which stands for the expected number of iterations before halting, of this two-dimensional SR scheme is the corresponding A/R constant $C$ in Algorithm 2.1. Hence,if we can find an appropriate bivariate envelope with a lower and uniformly bounded $C$, then this method is more suitable than the DR method used by Devroye [2009], as only one rejection procedure is involved within the entire simulation instead of two.

## 3 SIMULATION SCHEME FOR ETS DISTRIBUTION

Based on the two-dimensional SR framework in Algorithm 2.1, we design an efficient simulation algorithm to sample the ETS distributions with uniformly bounded complexity. First of all, let us define $\text{Erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ as the error function, $\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt$ as the gamma function, and denote $\mathcal{N}(\mu, \sigma^2, \text{lb} = 0, \text{ub} = \pi)$[3] as the truncated normal distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 \in \mathbb{R}^+$ within the domain $[0, \pi]$. The details of the new simulation scheme for ETS distributions is provided in Algorithm 3.1.

Proof. According to (2) and (4), for

$$X = \lambda S_{\alpha,\lambda}, \tag{6}$$

[3] The abbreviation "lb"stands for the lower bound, and "ub"stands for the upper bound.

**ALGORITHM 3.1:** Two-Dimensional SR Algorithm for $S_{\alpha,\lambda}$

(1) **set** $R = \mathrm{Erf}(\sqrt{\alpha(1-\alpha)\lambda^\alpha \pi^2/2})$, $C_1 = \frac{\Gamma(\alpha\lambda^\alpha)e^{\alpha\lambda^\alpha-1}}{(\alpha\lambda^\alpha)^{\lambda^\alpha}}\left(\frac{\alpha}{1-\alpha}+\alpha\lambda^\alpha\right)^{\lambda^\alpha(1-\alpha)+1}$,
$C_2 = \frac{\Gamma((1-\alpha)\lambda^\alpha+1)e^{(1-\alpha)\lambda^\alpha}}{((1-a)\lambda^\alpha)^{(1-\alpha)\lambda^\alpha}}$, $C_3 = \frac{\Gamma(\alpha\lambda^\alpha+1)e^{\alpha\lambda^\alpha-1}(\alpha\lambda^\alpha)^{-\alpha\lambda^\alpha}}{\sqrt{2\pi\alpha(1-\alpha)\lambda^\alpha}(1+1/[(1-\alpha)\lambda^\alpha])^{-1-(1-\alpha)\lambda^\alpha}}$,
$C_4 = \frac{\Gamma((1-\alpha)\lambda^\alpha+1)e^{(1-\alpha)\lambda^\alpha}}{\sqrt{2\pi\alpha(1-\alpha)\lambda^\alpha}((1-\alpha)\lambda^\alpha)^{(1-\alpha)\lambda^\alpha}}$
(2) **if** $(C_1 = \min\{C_1, C_2, C_3, C_4\})$ {
(3) **repeat** {
(4) **sample** $U \sim \mathcal{U}[0,\pi]$, $X \sim \Gamma(\alpha\lambda^\alpha, 1)$, $V \sim \mathcal{U}[0,1]$; **set** $S = X/\lambda$
(5) **if** $\left(V \le \frac{\alpha e^{\lambda^\alpha}\Gamma(\alpha\lambda^\alpha)}{1-\alpha}B(U)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}X^{-\frac{\alpha}{1-\alpha}-\alpha\lambda^\alpha}e^{-B(U)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}X^{-\frac{\alpha}{1-\alpha}}}/C_1\right)$ **break**
(6) }
(7) }
(8) **if** $(C_2 = \min\{C_1, C_2, C_3, C_4\})$ {
(9) **repeat**{
(10) **sample** $U \sim \mathcal{U}[0,\pi]$, $Z \sim \Gamma\left((1-\alpha)\lambda^\alpha+1, 1\right)$, $V \sim \mathcal{U}[0,1]$; **set** $S = B(U)^{\frac{1}{\alpha}}Z^{-\frac{1-\alpha}{\alpha}}$
(11) **if** $\left(V \le e^{\lambda^\alpha}\Gamma((1-\alpha)\lambda^\alpha+1)Z^{-(1-\alpha)\lambda^\alpha}e^{-\lambda B(U)^{\frac{1}{\alpha}}Z^{-\frac{1-\alpha}{\alpha}}}/C_2\right)$ **break**
(12) }
(13) }
(14) **if** $(C_3 = \min\{C_1, C_2, C_3, C_4\})$ {
(15) **repeat**{
(16) **sample** $U \sim \mathcal{N}\left(\mu = 0, \sigma^2 = [\alpha(1-\alpha)\lambda^\alpha]^{-1}, \mathrm{lb} = 0, \mathrm{ub} = \pi\right)$
(17) **sample** $X \sim \Gamma(\alpha\lambda^\alpha, 1)$, $V \sim \mathcal{U}[0,1]$; **set** $S = X/\lambda$
(18) **if** $\left(V \le \frac{R\alpha e^{\lambda^\alpha}\Gamma(\alpha\lambda^\alpha)\lambda^{\frac{\alpha}{1-\alpha}}B(U)^{\frac{1}{1-\alpha}}}{C_3(1-\alpha)\sqrt{2\pi\alpha(1-\alpha)\lambda^\alpha}X^{\frac{\alpha}{1-\alpha}+\alpha\lambda^\alpha}}e^{-\left(\lambda B(u)^{\frac{1}{\alpha}}X^{-1}\right)^{\frac{\alpha}{1-\alpha}}+\frac{\alpha(1-\alpha)\lambda^\alpha U^2}{2}}\right)$ **break**
(19) }
(20) }
(21) **if** $(C_4 = \min\{C_1, C_2, C_3, C_4\})$ {
(22) **repeat**{
(23) **sample** $U \sim \mathcal{N}\left(\mu = 0, \sigma^2 = [\alpha(1-\alpha)\lambda^\alpha]^{-1}, \mathrm{lb} = 0, \mathrm{ub} = \pi\right)$
(24) **sample** $Z \sim \Gamma\left((1-\alpha)\lambda^\alpha+1, 1\right)$, $V \sim \mathcal{U}[0,1]$; **set** $S = B(U)^{\frac{1}{\alpha}}Z^{-\frac{1-\alpha}{\alpha}}$
(25) **if** $\left(V \le \frac{Re^{\lambda^\alpha}\Gamma((1-\alpha)\lambda^\alpha+1)}{C_4\sqrt{2\pi\alpha(1-\alpha)\lambda^\alpha}Z^{(1-\alpha)\lambda^\alpha}}e^{-\lambda B(U)^{\frac{1}{\alpha}}Z^{-\frac{1-\alpha}{\alpha}}+\frac{\alpha(1-\alpha)\lambda^\alpha U^2}{2}}\right)$ **break**
(26) }
(27) }
(28) **return** $S$

the density of the random variable $X$ is specified by

$$f_X(x) = \frac{1}{\pi}\int_0^\pi \frac{\alpha e^{\lambda^\alpha}}{1-\alpha}B(u)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}x^{-\frac{1}{1-\alpha}}e^{-B(u)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}x^{-\frac{\alpha}{1-\alpha}}-x}\mathrm{d}u, \qquad x \in \mathbb{R}^+,$$

which is the marginal density of the bivariate variable $(X, U)$ on $[0,\infty)\times[0,\pi]$ with density

$$f(x,u) = \frac{\alpha e^{\lambda^\alpha}}{\pi(1-\alpha)}B(u)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}x^{-\frac{1}{1-\alpha}}\exp\left(-B(u)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}x^{-\frac{\alpha}{1-\alpha}}-x\right). \tag{7}$$

To sample $S_{\alpha,\lambda}$, first we sample $(X,U)$ by applying the two-dimensional SR scheme in (2.1), and then return

$$S_{\alpha,\lambda} = X/\lambda.$$

To simulate $(X,U)$ with density (7), we could choose a gamma-uniform bivariate envelope $(X',U')$ on $[0,\infty)\times[0,\pi]$ with density

$$g(x,u) \;=\; \frac{1}{\pi}\frac{1}{\Gamma(m)}x^{m-1}e^{-x}, \tag{8}$$

for some $m\in\mathbb{R}^+$. Given the density function $f(x,u)$ for $(X,U)$ in (7) and $g(x,u)$ for $(X',U')$ in (8), we have

$$\begin{aligned}
\frac{f(x,u)}{g(x,u)} &= \frac{\alpha e^{\lambda^\alpha}\Gamma(m)}{1-\alpha}B(u)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}x^{-\frac{\alpha}{1-\alpha}-m}\exp\left(-B(u)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}x^{-\frac{\alpha}{1-\alpha}}\right)\\
&\le \frac{\alpha e^{\lambda^\alpha}\Gamma(m)}{1-\alpha}B(u)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}\left[\frac{\frac{\alpha}{1-\alpha}B(u)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}}{\frac{\alpha}{1-\alpha}+m}\right]^{-\frac{(1-\alpha)m+\alpha}{\alpha}}\\
&\quad \exp\left(-B(u)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}\left[\frac{\alpha B(u)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}}{\alpha+(1-\alpha)m}\right]^{-1}\right)\\
&= \left(\frac{\alpha}{1-\alpha}\right)^{-\frac{m(1-\alpha)}{\alpha}}\lambda^{-m}\Gamma(m)\left(\frac{\alpha}{1-\alpha}+m\right)^{\frac{m(1-\alpha)+\alpha}{\alpha}}e^{-\frac{m(1-\alpha)+\alpha}{\alpha}+\lambda^\alpha}B(u)^{-\frac{m}{\alpha}}\\
&\le \left(\frac{\alpha}{1-\alpha}\right)^{-\frac{m(1-\alpha)}{\alpha}}\lambda^{-m}\Gamma(m)\left(\frac{\alpha}{1-\alpha}+m\right)^{\frac{m(1-\alpha)+\alpha}{\alpha}}e^{-\frac{m(1-\alpha)+\alpha}{\alpha}+\lambda^\alpha}B(0)^{-\frac{m}{\alpha}}\\
&= \left(\frac{\alpha}{1-\alpha}\right)^{-\frac{m(1-\alpha)}{\alpha}}\lambda^{-m}\Gamma(m)\left(\frac{\alpha}{1-\alpha}+m\right)^{\frac{m(1-\alpha)+\alpha}{\alpha}}e^{-\frac{m(1-\alpha)+\alpha}{\alpha}+\lambda^\alpha}\left[(1-\alpha)^{1-\alpha}\alpha^\alpha\right]^{-\frac{m}{\alpha}}\\
&= C_1(\alpha,\lambda;m),
\end{aligned}$$

where $B(u)$ is a monotone increasing function with

$$\min_{0\le u\le\infty}\{B(u)\} \;=\; B(0) = (1-\alpha)^{1-\alpha}\alpha^\alpha. \tag{9}$$

The A/R constant $C_1(\alpha,\lambda;m)$ can be further minimized over $m$. The optimal value $m^*$ satisfies

$$\frac{\alpha}{1-\alpha}\psi^{(0)}(m^*)+\ln\left(\frac{\alpha}{1-\alpha}+m^*\right) \;=\; \ln\left(\alpha^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}\right), \quad \text{for } \psi^{(0)}(m)=\frac{\mathrm{d}\Gamma(m)}{\mathrm{d}m}. \tag{10}$$

Hence, by approximating the LHS of (10), the optimal rate $m^*$ for the gamma-distributed envelope is chosen by setting $m^*=\alpha\lambda^\alpha$. The A/R decision therefore follows

$$V \;\le\; \frac{f(X',U')}{C_1(\alpha,\lambda)g(X',U')},$$

with

$$C_1(\alpha,\lambda) \;=\; C_1\left(\alpha,\lambda;\alpha\lambda^\alpha\right) = (\alpha\lambda^\alpha)^{-\lambda^\alpha}\,e^{\alpha\lambda^\alpha-1}\Gamma\left(\alpha\lambda^\alpha\right)\left(\frac{\alpha}{1-\alpha}+\alpha\lambda^\alpha\right)^{\lambda^\alpha(1-\alpha)+1}, \tag{11}$$

where $C_1(\alpha,\lambda)$ is the associated A/R constant to sample $(X,U)$ via a gamma-uniform bivariate envelope $(X',U')$. Instead of this gamma-uniform bivariate envelope, one could use a gamma and truncated-normal bivariate envelope $(\bar{X},\bar{U})$ on $[0,\infty)\times[0,\pi]$ with associated density of the form

$$h(x,u) \;=\; \frac{x^{\alpha\lambda^\alpha-1}e^{-x}}{\Gamma\left(\alpha\lambda^\alpha\right)}\frac{\sqrt{2\alpha(1-\alpha)\lambda^\alpha}/\sqrt{\pi}}{\mathrm{Erf}\left(\pi\sqrt{\alpha(1-\alpha)\lambda^\alpha/2}\right)}e^{-\frac{\alpha(1-\alpha)\lambda^\alpha u^2}{2}}, \tag{12}$$

to implement the two-dimensional SR scheme. We consider a new envelope $\left(\bar{X}, \bar{U}\right)$ such that

$$\bar{X} \sim \Gamma\left(\alpha\lambda^{\alpha},\ 1\right), \qquad \bar{U} \sim \mathcal{N}\left(\mu = 0,\ \sigma^2 = \frac{1}{\alpha(1-\alpha)\lambda^{\alpha}},\ \text{lb} = 0,\ \text{ub} = \pi\right),$$

which is a truncated-normal random variable with mean $\mu = 0$ and variance $\sigma^2 = \frac{1}{\alpha(1-\alpha)\lambda^{\alpha}}$ within the domain $[0, \pi]$. Given the joint density of $(X, U)$ in (7) and the joint density of $\left(\bar{X}, \bar{U}\right)$ in (12), first, by maximizing $f(x,u)/g(x,u)$ with respect to $x$, we have

$$\begin{aligned}\frac{f(x,u)}{h(x,u)} &\leq \frac{\text{Erf}\left(\pi\sqrt{\alpha(1-\alpha)\lambda^{\alpha}/2}\right)\alpha e^{\lambda^{\alpha}}\Gamma(\alpha\lambda^{\alpha})}{(1-\alpha)\sqrt{2\pi\alpha(1-\alpha)\lambda^{\alpha}}}\lambda^{-\alpha\lambda^{\alpha}}B(u)^{-\lambda^{\alpha}}e^{\frac{\alpha(1-\alpha)\lambda^{\alpha}u^2}{2}} \\ &\quad \times\left(1+(1-\alpha)\lambda^{\alpha}\right)^{1+(1-\alpha)\lambda^{\alpha}}e^{-(1+(1-\alpha)\lambda^{\alpha})}.\end{aligned}$$

According to Devroye [2009], we have the inequality

$$B(u)^{-\lambda^{\alpha}} \leq B(0)^{-\lambda^{\alpha}}e^{-\frac{\alpha(1-\alpha)\lambda^{\alpha}u^2}{2}} = \left[\alpha^{\alpha}(1-\alpha)^{1-\alpha}\right]^{-\lambda^{\alpha}}e^{-\frac{\alpha(1-\alpha)\lambda^{\alpha}u^2}{2}}. \tag{13}$$

Hence, by (13), we then have

$$\begin{aligned}\frac{f(x,u)}{h(x,u)} &\leq \frac{\text{Erf}\left(\pi\sqrt{\alpha(1-\alpha)\lambda^{\alpha}/2}\right)\Gamma(\alpha\lambda^{\alpha}+1)e^{-1+\alpha\lambda^{\alpha}}(\alpha\lambda^{\alpha})^{-\alpha\lambda^{\alpha}}(1-\alpha)^{-1-(1-\alpha)\lambda^{\alpha}}}{\sqrt{2\pi\alpha(1-\alpha)\lambda^{\alpha}}\left(\frac{1}{\lambda^{\alpha}}+(1-\alpha)\right)^{-1-(1-\alpha)\lambda^{\alpha}}} \\ &\leq \frac{\Gamma(\alpha\lambda^{\alpha}+1)e^{-1+\alpha\lambda^{\alpha}}(\alpha\lambda^{\alpha})^{-\alpha\lambda^{\alpha}}}{\sqrt{2\pi\alpha(1-\alpha)\lambda^{\alpha}}}\left(1+\frac{1}{(1-\alpha)\lambda^{\alpha}}\right)^{1+(1-\alpha)\lambda^{\alpha}} \\ &= C_3(\alpha,\lambda),\end{aligned} \tag{14}$$

where $C_3(\alpha,\lambda)$ is the associated A/R constant to sample $(X,U)$ via a gamma and truncated-normal bivariate envelope $(\bar{X},\bar{U})$. Given these two methodologies, one could set $S_{\alpha,\lambda} = X/\lambda$ to obtain the realization of $S_{\alpha,\lambda}$ once $X$ has been generated.

Besides $X = \lambda S_{\alpha,\lambda}$, we could use an alternative transformation to sample $S_{\alpha,\lambda}$ by setting

$$Z = B(U)^{\frac{1}{1-\alpha}}S_{\alpha,\lambda}^{-\frac{\alpha}{1-\alpha}}. \tag{15}$$

According to (5), by changing the variables of the joint distribution function from $(S,U)$ to $(Z,U)$, the bivariate density function of $(Z,U)$ in $(z,u)$ on $[0,\infty)\times[0,\pi]$ is of the form

$$f(z,u) = \frac{e^{\lambda^{\alpha}}}{\pi}\exp\left(-z-\lambda B(u)^{\frac{1}{\alpha}}z^{-\frac{1-\alpha}{\alpha}}\right). \tag{16}$$

To sample $S_{\alpha,\lambda}$, we could sample $(Z,U)$ first and then return

$$S_{\alpha,\lambda} = B(U)^{\frac{1}{\alpha}}Z^{-\frac{1-\alpha}{\alpha}}.$$

To simulate $(Z,U)$ with density (16), we choose an envelope $(Z',U')$ on $[0,\infty)\times[0,\pi]$ with joint density function

$$g(z,u) = \frac{1}{\pi}\frac{z^r e^{-z}}{\Gamma(r+1)}.$$

According to (9), we have

$$\begin{aligned}\frac{f(z,u)}{g(z,u)} &\leq e^{\lambda^{\alpha}}\Gamma(r+1)z^{-r}\exp\left(-\lambda\alpha(1-\alpha)^{\frac{1-\alpha}{\alpha}}z^{-\frac{1-\alpha}{\alpha}}\right) \\ &\leq \left(\frac{\alpha r}{(1-\alpha)\lambda}\right)^{\frac{r\alpha}{1-\alpha}}e^{-\frac{r\alpha}{1-\alpha}+\lambda^{\alpha}}\Gamma(r+1)\left[(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\right]^{-r} = C_2(\alpha,\lambda;r),\end{aligned}$$

where $C_2(\alpha,\lambda;r)$ can be minimized over $r$. The optimal value $r^*$ satisfies

$$\psi^{(0)}(r^*+1) \;=\; \frac{\alpha}{1-\alpha}\ln\left(\frac{\lambda(1-\alpha)^{\frac{1}{\alpha}}}{r^*}\right), \quad \text{for } \psi^{(0)}(r)=\frac{\mathrm{d}\Gamma(r)}{\mathrm{d}r}. \tag{17}$$

By approximating the LHS of (17), the optimal rate $r^*$ is chosen by setting $r^*=(1-\alpha)\lambda^\alpha$. Hence, the associated A/R constant with $r^*$ is given by

$$C_2(\alpha,\lambda) \;=\; C_2\Big(\alpha,\lambda;(1-\alpha)\lambda^\alpha\Big) \;=\; \Gamma\Big((1-\alpha)\lambda^\alpha+1\Big)e^{(1-\alpha)\lambda^\alpha}\left((1-\alpha)\lambda^\alpha\right)^{-(1-\alpha)\lambda^\alpha}, \tag{18}$$

where $C_2(\alpha,\lambda)$ is the associated A/R constant to sample $(Z,U)$ via a gamma-uniform bivariate envelope $(Z',U')$. Similarly, one could also consider a gamma and truncated-normal bivariate envelope $\left(\bar{Z},\bar{U}\right)$ for $(Z,U)$ on $[0,\infty)\times[0,\pi]$ with density (16) such that

$$\bar{Z} \;\sim\; \Gamma\Big((1-\alpha)\lambda^\alpha+1,\;1\Big), \qquad \bar{U} \;\sim\; \mathcal{N}\left(\mu=0,\;\sigma^2=\frac{1}{\alpha(1-\alpha)\lambda^\alpha},\;\mathrm{lb}=0,\;\mathrm{ub}=\pi\right).$$

The joint density is given as

$$h(z,u) \;=\; \frac{z^{(1-\alpha)\lambda^\alpha}e^{-z}}{\Gamma((1-\alpha)\lambda^\alpha+1)}\frac{\sqrt{2\alpha(1-\alpha)\lambda^\alpha}/\sqrt{\pi}}{\mathrm{Erf}\left(\pi\sqrt{\alpha(1-\alpha)\lambda^\alpha/2}\right)}e^{-\frac{\alpha(1-\alpha)\lambda^\alpha u^2}{2}}.$$

Then, by maximizing $f(z,u)/h(z,u)$ with respect to $z$ and applying inequality (13), we have

$$\begin{aligned}
\frac{f(z,u)}{h(z,u)} \;&=\; \frac{\mathrm{Erf}\left(\pi\sqrt{\alpha(1-\alpha)\lambda^\alpha/2}\right)e^{\lambda^\alpha}\Gamma((1-\alpha)\lambda^\alpha+1)}{\sqrt{2\pi\alpha(1-\alpha)\lambda^\alpha}}z^{-(1-\alpha)\lambda^\alpha} \\
&\quad \exp\left(-\lambda B(u)^{\frac{1}{\alpha}}z^{-\frac{1-\alpha}{\alpha}}+\frac{\alpha(1-\alpha)\lambda^\alpha u^2}{2}\right) \\
&\le\; \frac{\mathrm{Erf}\left(\pi\sqrt{\alpha(1-\alpha)\lambda^\alpha/2}\right)e^{(1-\alpha)\lambda^\alpha}\Gamma((1-\alpha)\lambda^\alpha+1)}{\sqrt{2\pi\alpha(1-\alpha)\lambda^\alpha}}\lambda^{-\alpha(1-\alpha)\lambda^\alpha}\alpha^{\alpha\lambda^\alpha}B(u)^{-\lambda^\alpha}e^{\frac{\alpha(1-\alpha)\lambda^\alpha u^2}{2}} \\
&\le\; \frac{\mathrm{Erf}\left(\pi\sqrt{\alpha(1-\alpha)\lambda^\alpha/2}\right)e^{(1-\alpha)\lambda^\alpha}\Gamma((1-\alpha)\lambda^\alpha+1)}{\sqrt{2\pi\alpha(1-\alpha)\lambda^\alpha}}(1-\alpha)^{-(1-\alpha)\lambda^\alpha}\lambda^{-\alpha(1-\alpha)\lambda^\alpha} \\
&\le\; \frac{\Gamma((1-\alpha)\lambda^\alpha+1)}{\sqrt{2\pi\alpha(1-\alpha)\lambda^\alpha}}\left((1-\alpha)\lambda^\alpha\right)^{-(1-\alpha)\lambda^\alpha}e^{(1-\alpha)\lambda^\alpha} \;=\; C_4(\alpha,\lambda),
\end{aligned} \tag{19}$$

where $C_4(\alpha,\lambda)$ is the associated A/R constant to sample $(Z,U)$ via a gamma and truncated-normal bivariate envelope $(\bar{Z},\bar{U})$. Given these two methodologies, one could set $S_{\alpha,\lambda} = B(U)^{\frac{1}{\alpha}}Z^{-\frac{1-\alpha}{\alpha}}$ to obtain the realization of $S_{\alpha,\lambda}$ once $X$ has been generated.

When specifying the input of $(\alpha,\lambda)$, to sample $S_{\alpha,\lambda}$, these four two-dimensional SR schemes will lead to different expected complexities, namely $C_1(\alpha,\lambda), C_2(\alpha,\lambda), C_3(\alpha,\lambda), C_4(\alpha,\lambda)$. Therefore, the most efficient strategy to sample $S_{\alpha,\lambda}$ is to choose the one with the smallest highest acceptance rate, to implement the corresponding two-dimensional SR procedure, which leads to Algorithm 3.1. And the overall complexity therefore would be formidable by $C(\alpha,\lambda)$, where

$$C(\alpha,\lambda) \;=\; \min_{i=1,2,3,4}\{C_i(\alpha,\lambda):\;(\alpha,\lambda)\in(0,1)\times(0,\infty)\}. \tag{20}$$

□

Given the complexity $C(\alpha,\lambda)$ in (20) for Algorithm 3.1, we conclude the following result.

Theorem 3.1. *The complexity $C(\alpha,\lambda)$ in (20) for Algorithm 3.1 is uniformly bounded. In particular, we have*

$$\sup_{\alpha\in(0,1),\lambda\geq 0} C(\alpha,\lambda) \leq 4.2154.$$

Proof. According to Corollary 1.2 in Batir [2008], the following inequality,

$$\Gamma(x+1) < \sqrt{2\pi}x^x e^{-x}\sqrt{x+\frac{1}{2}},$$

holds for $x \geq 0$, and we then have the following results:

$$\begin{aligned}
C_1(\alpha,\lambda) &< \sqrt{2\pi\alpha\lambda^\alpha+\pi}\left(1+\frac{1}{(1-\alpha)\lambda^\alpha}\right) = \bar{C}_1(\alpha,\lambda),\\
C_2(\alpha,\lambda) &< \sqrt{2\pi(1-\alpha)\lambda^\alpha+\pi} = \bar{C}_2(\alpha,\lambda),\\
C_3(\alpha,\lambda) &< \sqrt{\frac{1}{1-\alpha}+\frac{1}{2\alpha(1-\alpha)\lambda^\alpha}}\left(1+\frac{1}{(1-\alpha)\lambda^\alpha}\right) = \bar{C}_3(\alpha,\lambda),\\
C_4(\alpha,\lambda) &< \sqrt{\frac{1}{\alpha}+\frac{1}{2\alpha(1-\alpha)\lambda^\alpha}} = \bar{C}_4(\alpha,\lambda).
\end{aligned}$$

Hence, for any combination of $(\alpha,\lambda)\in(0,1)\times(0,\infty)$, we have

$$\begin{aligned}
C(\alpha,\lambda) &= \min_{i=1,2,3,4}\{C_i(\alpha,\lambda):\ (\alpha,\lambda)\in(0,1)\times(0,\infty)\}\\
&< \min_{i=1,2,3,4}\{\bar{C}_i(\alpha,\lambda):\ (\alpha,\lambda)\in(0,1)\times(0,\infty)\}\\
&= \bar{C}(\alpha,\lambda). \qquad (21)
\end{aligned}$$

To prove $C(\alpha,\lambda)$ is uniformly bounded over $(\alpha,\lambda)\in(0,1)\times(0,\infty)$, it suffices to prove $\bar{C}(\alpha,\lambda)$ is uniformly bounded over $(\alpha,\lambda)\in(0,1)\times(0,\infty)$. We notice that the following inequalities hold:

$$\begin{aligned}
\min\left\{\sqrt{2\pi\alpha\lambda^\alpha+\pi},\ \sqrt{\frac{1}{1-\alpha}+\frac{1}{2\alpha(1-\alpha)\lambda^\alpha}}\right\} &\leq \sqrt{\frac{1}{1-\alpha}+\pi},\\
\min\left\{\sqrt{2\pi(1-\alpha)\lambda^\alpha+\pi},\ \sqrt{\frac{1}{\alpha}+\frac{1}{2\alpha(1-\alpha)\lambda^\alpha}}\right\} &\leq \sqrt{\frac{1}{\alpha}+\pi},
\end{aligned}$$

for any arbitrary $(\alpha,\lambda)\in(0,1)\times(0,\infty)$,[4] which indicate that

$$\begin{aligned}
\min\{\bar{C}_1(\alpha,\lambda),\ \bar{C}_3(\alpha,\lambda)\} &\leq \sqrt{\frac{1}{1-\alpha}+\pi}\left(1+\frac{1}{(1-\alpha)\lambda^\alpha}\right),\\
\min\{\bar{C}_2(\alpha,\lambda),\ \bar{C}_4(\alpha,\lambda)\} &\leq \sqrt{\frac{1}{\alpha}+\pi}.
\end{aligned}$$

Hence, we have

$$\bar{C}(\alpha,\lambda) \leq \min\left\{\bar{C}_2(\alpha,\lambda),\ \sqrt{\frac{1}{\alpha}+\pi},\ \sqrt{\frac{1}{1-\alpha}+\pi}\left(1+\frac{1}{(1-\alpha)\lambda^\alpha}\right)\right\}. \qquad (22)$$

[4]The supreme value of the function defined by the minimum of an increasing and decreasing function is at the point when the two functions meet.

Note that since $\bar{C}_2(\alpha,\lambda)$ goes to $\sqrt{3\pi}$ when both $\alpha,\lambda \to 0$, this $\bar{C}_2(\alpha,\lambda)$ in (22) will prevent explosion when both $\alpha,\lambda \to 0$.

First, for the case $\alpha \in [\frac{1}{2},1)$, since $\sqrt{\frac{1}{\alpha}+\pi}$ is decreasing and bounded, we have

$$\bar{C}(\alpha,\lambda) \;\le\; \sqrt{\frac{1}{\alpha}+\pi} \;\le\; \sqrt{2+\pi} \;\approx\; 2.2675.$$

For the case $\alpha \in (0,\frac{1}{2})$, we have

$$\begin{aligned}
\bar{C}(\alpha,\lambda) &\le \min\left\{\sqrt{2\pi(1-\alpha)\lambda^\alpha+\pi},\ \sqrt{\frac{1}{1-\alpha}+\pi}\left(1+\frac{1}{(1-\alpha)\lambda^\alpha}\right)\right\}\\
&\le \min\left\{\sqrt{2\pi\lambda^\alpha+\pi},\ \sqrt{2+\pi}\left(1+\frac{2}{\lambda^\alpha}\right)\right\}\\
&\le \max_{(\alpha,\lambda)\in(0,1)\times(0,\infty)}\left\{\min\left\{\sqrt{2\pi\lambda^\alpha+\pi},\ \sqrt{2+\pi}\left(1+\frac{2}{\lambda^\alpha}\right)\right\}\right\} \;\approx\; 4.2154,
\end{aligned}$$

where this supreme value is obtained at $\lambda^\alpha \approx 2.3281$—that is, when

$$\sqrt{2\pi\lambda^\alpha+\pi} \;=\; \sqrt{2+\pi}\left(1+\frac{2}{\lambda^\alpha}\right).$$

This is because one of the functions is increasing in $\lambda^\alpha$ and the other function is decreasing in $\lambda^\alpha$, and the maximum of the minimum of these two function over $(\alpha,\lambda)\in(0,1)\times(0,\infty)$ is at the point when these two functions are equal.

Hence, by (21), we have

$$C(\alpha,\lambda) \;\le\; \bar{C}(\alpha,\lambda) \;\le\; 4.2154\times\mathbf{1}_{\{0<\alpha<1/2\}} + 2.2675\times\mathbf{1}_{\{1/2\le\alpha<1\}} \;\le\; 4.2154,$$

which clearly implies that $C(\alpha,\lambda)$ is uniformly bounded by 4.2154 over $(0,1)\times(0,\infty)$. □

This $C(\alpha,\lambda)$ is uniformly bounded by 4.2154 over all combinations of the parameters. When the stability parameter $\alpha$ is between 1/2 and 1, the upper bound can be reduced to 2.2685. In principle, the uniform bound provided in Theorem 3.1 is the bound for $\bar{C}(\alpha,\lambda)$, which is the upper bound of $C(\alpha,\lambda)$, whereas the actual bound of $C(\alpha,\lambda)$ is much smaller than this uniform bound.

*Remark 3.1.* Given $C(\alpha,\lambda)$ in (20), when holding $\lambda$ fixed, we have

$$\begin{aligned}
\lim_{\alpha\to0} C(\alpha,\lambda) &= \lim_{\alpha\to0} C_1(\alpha,\lambda) = \lim_{\alpha\to0} e^{-1}\left(1+\frac{1}{(1-\alpha)\lambda^\alpha}\right)^{\lambda^\alpha+1} = \frac{4}{e},\\
\lim_{\alpha\to1} C(\alpha,\lambda) &= \lim_{\alpha\to1} C_2(\alpha,\lambda) = \lim_{x\to0}\Gamma(x+1)x^{-x}e^x = 1,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\lambda\to0} C(\alpha,\lambda) &= \lim_{\lambda\to0} C_2(\alpha,\lambda) = \lim_{x\to0}\Gamma(x+1)x^{-x}e^x = 1,\\
\lim_{\lambda\to\infty} C(\alpha,\lambda) &\le \lim_{\lambda\to\infty}\bar{C}(\alpha,\lambda) = \min_{\alpha\in(0,1)}\left\{\sqrt{1/\alpha},\sqrt{1/1-\alpha}\right\} \le \sqrt{2},
\end{aligned}$$

while holding $\alpha$ fixed. Figure 2 shows the value of $C(\alpha,\lambda)$ for various values of $\alpha$ and $\lambda$. The calculated maximum it attains for those values is about 2.5. This actual bound for $C(\alpha,\lambda)$ we observed is much smaller than the one we discovered in Theorem 3.1.

In fact, this $C(\alpha,\lambda)$ is indeed the complexity of the scheme that optimally combines the four two-dimensional SR algorithms with different envelopes and implements the most efficient algorithm by choosing the one with the smallest A/R constant to sample the ETS random variable $S_{\alpha,\lambda}$. The overall complexity of Algorithm 3.1 is $C(\alpha,\lambda)$ in (20), which, according to Theorem 3.1, is uniformly

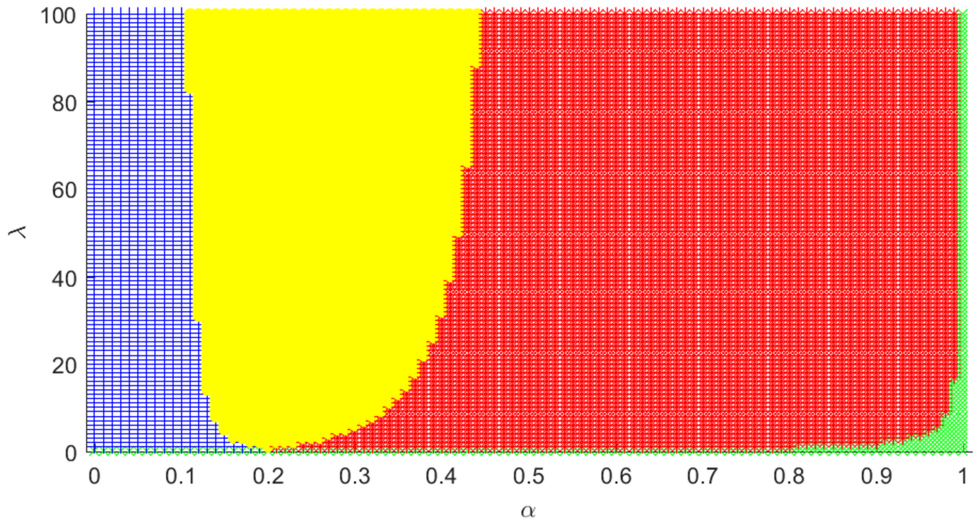


Fig. 1. Algorithm active regions. The blue region represents $C(\alpha,\lambda) = C_1(\alpha,\lambda)$, the green region represents $C(\alpha,\lambda) = C_2(\alpha,\lambda)$, the yellow region represents $C(\alpha,\lambda) = C_3(\alpha,\lambda)$, and the red region represents $C(\alpha,\lambda) = C_4(\alpha,\lambda)$.

bounded by 4.2154. Apparently, the complexity is smaller than 8.1133, which is the complexity of the DR scheme [Devroye 2009], and the relevant numerical comparison tests of these algorithms will be illustrated in Section 5. Figure 1 represents the plot of the regions over $(0,1)\times(0,\infty)$ where each of the two-dimensional SR algorithms suggested in Algorithm 3.1 will be active. We can see that for $\alpha$ close to 0, the two-dimensional SR algorithm with $C_1(\alpha,\lambda)$ will be active, and for $\alpha$ close to 1, or $\lambda$ close to 0, the two-dimensional SR algorithm with $C_3(\alpha,\lambda)$ will be active. When $\lambda$ is getting large, depending on the size of $\alpha$, one of the two-dimensional SR algorithms with $C_2(\alpha,\lambda)$ and $C_4(\alpha,\lambda)$ will be active. The limits for $C(\alpha,\lambda)$ provided in Remark 3.1 clearly explain these facts.

## 4 SIMULATION FOR GTS DISTRIBUTION

Beside the ETS class, the GTS distribution is another interesting class of tilted stable distributions. The GTS, denoted by $G_{\alpha,\lambda,\nu}$, was first introduced by Barndorff-Nielsen and Shephard [2001], and its density is defined as

$$f(s) = \frac{s^{\nu}e^{-\lambda s}f_{\alpha}(s)}{G(\nu,\lambda)}, \qquad s\in\mathbb{R}^{+}, \tag{23}$$

where $\nu > 0$, and $G(\nu,\lambda) = \int_0^{\infty} y^{\nu}e^{-\lambda y}f_{\alpha}(y)\mathrm{d}y$, and $f_{\alpha}(\cdot)$ is the density of the positive stable variable $S_{\alpha}$ in (2). Based on the two-dimensional SR method and the gamma-uniform envelope and the gamma and truncated-normal envelope for the ETS, we can also develop a simulation scheme to sample the GTS variables. The details are provided in Algorithm 4.1.

Proof. Given $X = \lambda G_{\alpha,\lambda,\nu}$, the bivariate density of $(X,U)$ on $[0,\infty)\times[0,\pi]$ is given by

$$f(x,u) = \frac{\alpha\lambda^{-\nu}e^{\lambda^{\alpha}}}{G(\nu,\lambda)\pi(1-\alpha)}B(u)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}x^{-\frac{1}{1-\alpha}+\nu}\exp\left(-B(u)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}x^{-\frac{\alpha}{1-\alpha}}-x\right). \tag{24}$$

**ALGORITHM 4.1:** Scheme for Simulating GTS $G_{\alpha,\lambda,\nu}$

(1) **set** $R_1 = \text{Erf}\left(\sqrt{\alpha(1-\alpha)\lambda^\alpha\pi^2/2}\right)$, $R_2 = \text{Erf}\left(\pi\sqrt{(1-\alpha)(\alpha\lambda^\alpha-\nu)/2}\right)$

(2) **set** $G_1 = \frac{e^{\alpha\lambda^\alpha-1}}{(\alpha\lambda^\alpha)^{\lambda^\alpha}}\left(\frac{\alpha+\alpha(1-\alpha)\lambda^\alpha}{1-\alpha}\right)^{\lambda^\alpha(1-\alpha)+1}$, $G_2 = \frac{e^{(1-\alpha)\lambda^\alpha(1-\alpha)}\frac{(1-\alpha)\nu}{\alpha}\alpha^\nu}{(1-a)^{(1-\alpha)\lambda^\alpha}\lambda^{\alpha(1-\alpha)\lambda^\alpha}}$,
$G_3 = \frac{\lambda^{-\alpha^2\lambda^\alpha+\alpha}e^{-1+\alpha\lambda^\alpha}\alpha^{-\alpha\lambda^\alpha+1}(1-\alpha)^{-1-(1-\alpha)\lambda^\alpha}}{\sqrt{2\pi\alpha(1-\alpha)\lambda^\alpha(1/\lambda^\alpha+(1-\alpha))^{-1-(1-\alpha)\lambda^\alpha}}}$, $G_4 = \frac{e^{(1-\alpha)\lambda^\alpha}(1-\alpha)^{\frac{(1-\alpha)\nu}{\alpha}}\alpha^\nu}{\sqrt{2\pi\alpha(1-\alpha)\lambda^\alpha}\sqrt{1-\nu/[\alpha\lambda^\alpha]}((1-\alpha)\lambda^\alpha)^{(1-\alpha)\lambda^\alpha}}$,
$\bar{C}_1 = \Gamma(\alpha\lambda^\alpha+\nu)\lambda^{-\nu}G_1$, $\bar{C}_3 = \Gamma(\alpha\lambda^\alpha+\nu)\lambda^{-\nu}G_3$,
$\bar{C}_2 = \Gamma\left((1-\alpha)\lambda^\alpha-(1-\alpha)\nu/\alpha+1\right)G_2$, and $\bar{C}_4 = \Gamma\left((1-\alpha)\lambda^\alpha-(1-\alpha)\nu/\alpha+1\right)G_4$

(3) **if** $\left(\{\bar{C}_1 = \min\{\bar{C}_1,\bar{C}_2,\bar{C}_3,\bar{C}_4\}, \nu < \alpha\lambda^\alpha\}$ or $\{\bar{C}_1 = \min\{\bar{C}_1,\bar{C}_3\}, \nu \geq \alpha\lambda^\alpha\}\right)\{$
(4) **repeat**{
(5) **sample** $U \sim \mathcal{U}[0,\pi]$, $X \sim \Gamma(\alpha\lambda^\alpha+\nu,1)$, $V \sim \mathcal{U}[0,1]$; **set** $G = X/\lambda$
(6) **if** $\left(V \leq \frac{\alpha e^{\lambda^\alpha}}{G_1(1-\alpha)}B(U)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}X^{-\frac{\alpha}{1-\alpha}-\alpha\lambda^\alpha}e^{-B(U)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}X^{-\frac{\alpha}{1-\alpha}}}\right)$ **break**
(7) }
(8) }
(9) **if** $\left(\bar{C}_2 = \min\{\bar{C}_1,\bar{C}_2,\bar{C}_3,\bar{C}_4\}, \nu < \alpha\lambda^\alpha\right)\{$
(10) **repeat**{
(11) **sample** $U \sim \mathcal{U}(0,\pi)$
(12) **sample** $Z \sim \Gamma((1-\alpha)\lambda^\alpha-(1-\alpha)\nu/\alpha+1,1)$, $V \sim \mathcal{U}[0,1]$; **set** $G = B(U)^{\frac{1}{\alpha}}Z^{-\frac{1-\alpha}{\alpha}}$
(13) **if** $(V \leq e^{\lambda^\alpha}B(U)^{\frac{\nu}{\alpha}}Z^{-(1-\alpha)\lambda^\alpha}e^{-\lambda B(U)^{\frac{1}{\alpha}}Z^{-\frac{1-\alpha}{\alpha}}}/G_2)$ **break**
(14) }
(15) }
(16) **if** $\left(\bar{C}_3 = \min\{\bar{C}_1,\bar{C}_2,\bar{C}_3,\bar{C}_4\}, \nu < \alpha\lambda^\alpha\}$ or $\{\bar{C}_3 = \min\{\bar{C}_1,\bar{C}_3\}, \nu \geq \alpha\lambda^\alpha\}\right)\{$
(17) **repeat**{
(18) **sample** $U \sim \mathcal{N}\left(\mu=0, \sigma^2=[\alpha(1-\alpha)\lambda^\alpha]^{-1}, \text{lb}=0, \text{ub}=\pi\right)$
(19) **sample** $X \sim \Gamma(\alpha\lambda^\alpha+\nu,1)$, $V \sim \mathcal{U}[0,1]$; **set** $G = X/\lambda$
(20) **if** $\left(V \leq \frac{R_1\alpha e^{\lambda^\alpha}\lambda^{\frac{\alpha}{1-\alpha}}B(U)^{\frac{1}{1-\alpha}}}{G_3(1-\alpha)\sqrt{2\pi\alpha(1-\alpha)\lambda^\alpha}X^{\frac{\alpha}{1-\alpha}+\alpha\lambda^\alpha}}e^{-\left(\lambda B(u)^{\frac{1}{\alpha}}X^{-1}\right)^{\frac{\alpha}{1-\alpha}}+\frac{\alpha(1-\alpha)\lambda^\alpha U^2}{2}}\right)$ **break**
(21) }
(22) }
(23) **if** $\left(\bar{C}_4 = \min\{\bar{C}_1,\bar{C}_2,\bar{C}_3,\bar{C}_4\}, \nu < \alpha\lambda^\alpha\right)\{$
(24) **repeat**{
(25) **sample** $U \sim \mathcal{N}\left(\mu=0, \sigma^2=[(1-\alpha)(\alpha\lambda^\alpha-\nu)]^{-1}, \text{lb}=0, \text{ub}=\pi\right)$
(26) **sample** $Z \sim \Gamma((1-\alpha)\lambda^\alpha-(1-\alpha)\nu/\alpha+1,1)$, $V \sim \mathcal{U}[0,1]$; **set** $G = B(U)^{\frac{1}{\alpha}}Z^{-\frac{1-\alpha}{\alpha}}$
(27) **if** $\left(V \leq \frac{R_2}{G_4\sqrt{2\pi(1-\alpha)(\alpha\lambda^\alpha-\nu)}Z^{(1-\alpha)\lambda^\alpha}}e^{\lambda^\alpha}B(U)^{\frac{\nu}{\alpha}}e^{-\lambda B(U)^{\frac{1}{\alpha}}Z^{-\frac{1-\alpha}{\alpha}}+\frac{(1-\alpha)(\alpha\lambda^\alpha-\nu)U^2}{2}}\right)$ **break**
(28) }
(29) }
(30) **return** $G$

To generate $(X,U)$, we consider a bivariate envelope $(\tilde{X},\tilde{U})$ on $[0,\infty)\times[0,\pi]$ with $\tilde{X} \sim \Gamma(\alpha\lambda^\alpha+\nu,1)$ and $\tilde{U} \sim \mathcal{U}[0,\pi]$, and the joint density is given as

$$p(x,u) = \frac{1}{\pi}\frac{x^{\alpha\lambda^\alpha+\nu-1}e^{-x}}{\Gamma(\alpha\lambda^\alpha+\nu)}.$$

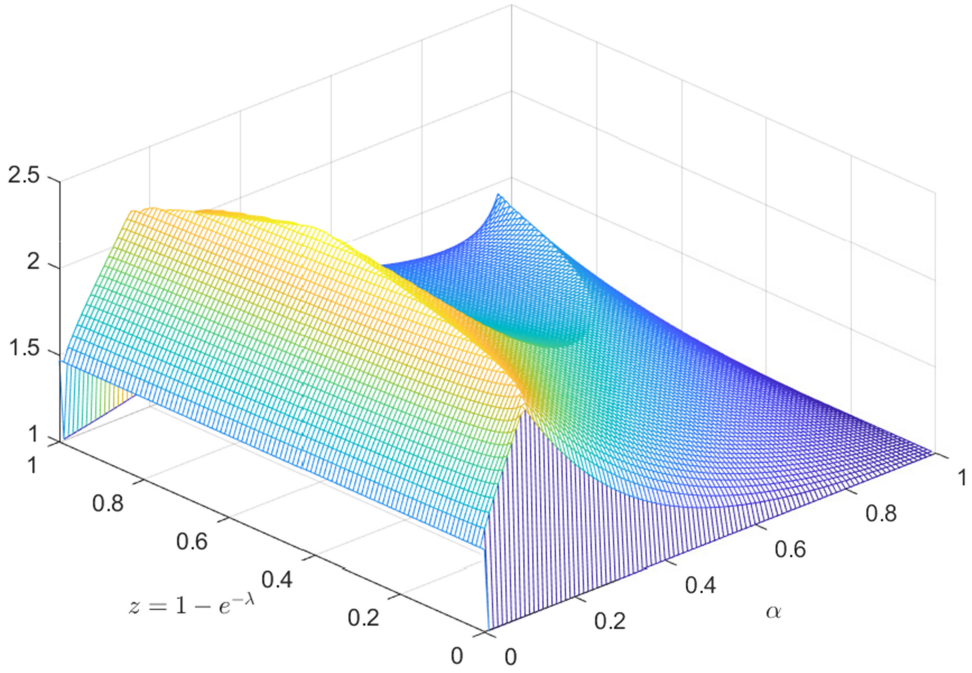


Fig. 2. The complexity of Algorithm 3.1 for $\alpha \in (0,1)$ and $\lambda := -\ln(1-z)$ with $z \in (0,1)$.

Hence, we have

$$\begin{aligned}\frac{f(x,u)}{p(x,u)} &= \frac{\lambda^{-\nu}\Gamma(\alpha\lambda^\alpha+\nu)}{G(\nu,\lambda)}\frac{\alpha e^{\lambda^\alpha}}{1-\alpha}B(u)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}x^{-\alpha\lambda^\alpha-\frac{\alpha}{1-\alpha}}\exp\left(-B(u)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}x^{-\frac{\alpha}{1-\alpha}}\right)\\ &= \frac{\lambda^{-\nu}\Gamma(\alpha\lambda^\alpha+\nu)}{G(\nu,\lambda)\Gamma(\alpha\lambda^\alpha)}\frac{\alpha e^{\lambda^\alpha}\Gamma(\alpha\lambda^\alpha)}{1-\alpha}B(u)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}x^{-\alpha\lambda^\alpha-\frac{\alpha}{1-\alpha}}\exp\left(-B(u)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}x^{-\frac{\alpha}{1-\alpha}}\right)\\ &\le \frac{C_1(\alpha,\lambda)\lambda^{-\nu}\Gamma(\alpha\lambda^\alpha+\nu)}{G(\nu,\lambda)\Gamma(\alpha\lambda^\alpha)} = \bar{C}_1,\end{aligned}$$

where $C_1(\alpha,\lambda)$ is defined in (11).

Alternatively, we consider a bivariate envelope $(\hat{X},\hat{U})$ on $[0,\infty)\times[0,\pi]$ such that

$$\hat{X} \sim \Gamma\left(\alpha\lambda^\alpha+\nu,\ 1\right), \qquad \hat{U} \sim \mathcal{N}\left(\mu=0,\ \sigma^2=\frac{1}{\alpha(1-\alpha)\lambda^\alpha},\ \text{lb}=0,\ \text{ub}=\pi\right),$$

and the joint density $(\hat{X},\hat{U})$ is given as

$$q(x,u) = \frac{x^{\alpha\lambda^\alpha+\nu-1}e^{-x}}{\Gamma(\alpha\lambda^\alpha+\nu)}\frac{\sqrt{2\alpha(1-\alpha)\lambda^\alpha}/\sqrt{\pi}}{\text{Erf}\left(\pi\sqrt{\alpha(1-\alpha)\lambda^\alpha/2}\right)}e^{-\frac{\alpha(1-\alpha)\lambda^\alpha u^2}{2}}.$$

According to the inequality (14), we have

$$\begin{aligned}\frac{f(x,u)}{q(x,u)} &= \frac{\lambda^{-\nu}\Gamma(\alpha\lambda^\alpha+\nu)\,\mathrm{Erf}\left(\pi\sqrt{\alpha(1-\alpha)\lambda^\alpha/2}\right)\alpha e^{\lambda^\alpha}}{G(\nu,\lambda)(1-\alpha)\sqrt{2\pi\alpha(1-\alpha)\lambda^\alpha}} B(u)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}x^{-\frac{\alpha}{1-\alpha}-\alpha\lambda^\alpha} \\ &\quad\times\exp\left(-B(u)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}x^{-\frac{\alpha}{1-\alpha}}+\frac{\alpha(1-\alpha)\lambda^\alpha u^2}{2}\right) \\ &= \frac{\lambda^{-\nu}\Gamma(\alpha\lambda^\alpha+\nu)}{G(\nu,\lambda)\Gamma(\alpha\lambda^\alpha)}\frac{\mathrm{Erf}\left(\pi\sqrt{\alpha(1-\alpha)\lambda^\alpha/2}\right)\alpha e^{\lambda^\alpha}\Gamma(\alpha\lambda^\alpha)}{(1-\alpha)\sqrt{2\pi\alpha(1-\alpha)\lambda^\alpha}} B(u)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}x^{-\frac{\alpha}{1-\alpha}-\alpha\lambda^\alpha} \\ &\quad\times\exp\left(-B(u)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}x^{-\frac{\alpha}{1-\alpha}}+\frac{\alpha(1-\alpha)\lambda^\alpha u^2}{2}\right) \\ &\le \frac{C_3(\alpha,\lambda)\lambda^{-\nu}\Gamma(\alpha\lambda^\alpha+\nu)}{G(\nu,\lambda)\Gamma(\alpha\lambda^\alpha)} = \bar{C}_3, \end{aligned} \tag{25}$$

which is the associated A/R constant.

For $\nu < \alpha\lambda^\alpha$, if we set

$$Z = B(U)^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{1-\alpha}}X^{-\frac{\alpha}{1-\alpha}},$$

then the joint density of $(Z,U)$ on $[0,\infty)\times[0,\pi]$ is

$$f(z,u) = \frac{e^{\lambda^\alpha}}{G(\nu,\lambda)\pi}B(u)^{\frac{\nu}{\alpha}}z^{-\frac{(1-\alpha)\nu}{\alpha}}\exp\left(-z-\lambda B(u)^{\frac{1}{\alpha}}z^{-\frac{1-\alpha}{\alpha}}\right). \tag{26}$$

Hence, to generate $(Z,U)$, we could choose a bivariate envelope $(\tilde{Z},\tilde{U})$ on $[0,\infty)\times[0,\pi]$ with density

$$\tilde{p}(z,u) = \frac{z^{(1-\alpha)\lambda^\alpha-\frac{(1-\alpha)\nu}{\alpha}}e^{-z}}{\pi\Gamma\left((1-\alpha)\lambda^\alpha-\frac{(1-\alpha)\nu}{\alpha}+1\right)},$$

and we have

$$\begin{aligned}\frac{f(z,u)}{\tilde{p}(z,u)} &= \frac{\Gamma\left((1-\alpha)\lambda^\alpha-\frac{(1-\alpha)\nu}{\alpha}+1\right)e^{\lambda^\alpha}}{G(\nu,\lambda)}B(u)^{\frac{\nu}{\alpha}}z^{-(1-\alpha)\lambda^\alpha}\exp\left(-\lambda B(u)^{\frac{1}{\alpha}}z^{-\frac{1-\alpha}{\alpha}}\right) \\ &\le \frac{\Gamma\left((1-\alpha)\lambda^\alpha-\frac{(1-\alpha)\nu}{\alpha}+1\right)e^{(1-\alpha)\lambda^\alpha}}{G(\nu,\lambda)}\lambda^{-\alpha(1-\alpha)\lambda^\alpha}\alpha^{\alpha\lambda^\alpha}\left[(1-\alpha)^{(1-\alpha)}\alpha^\alpha\right]^{-\frac{\alpha\lambda^\alpha-\nu}{\alpha}} \\ &\le \frac{C_2(\alpha,\lambda)\Gamma\left((1-\alpha)\lambda^\alpha-\frac{(1-\alpha)\nu}{\alpha}+1\right)\alpha^\nu(1-\alpha)^{\frac{(1-\alpha)\nu}{\alpha}}}{G(\nu,\lambda)\Gamma((1-\alpha)\lambda^\alpha+1)} = \bar{C}_2, \end{aligned} \tag{27}$$

where $C_2(\alpha,\lambda)$ is defined in (18).

Alternatively, we could choose a bivariate envelope $(\hat{Z},\hat{U})$ on $[0,\infty)\times[0,\pi]$ with density

$$\hat{q}(z,u) = \frac{z^{(1-\alpha)\lambda^\alpha-\frac{(1-\alpha)\nu}{\alpha}}e^{-z}}{\Gamma\left((1-\alpha)\lambda^\alpha-\frac{(1-\alpha)\nu}{\alpha}+1\right)}\frac{\sqrt{2}}{\mathrm{Erf}\left(\pi/\sqrt{2\sigma^2}\right)\sqrt{\pi\sigma^2}}e^{-\frac{u^2}{2\sigma^2}},$$

with $\sigma^2=[(1-\alpha)(\alpha\lambda^\alpha-\nu)]^{-1}$—that is, we have

$$\hat{Z} \sim \Gamma\left((1-\alpha)\lambda^\alpha-\frac{(1-\alpha)\nu}{\alpha}+1,\ 1\right),$$

$$\hat{U} \sim \mathcal{N}\left(\mu=0,\ \sigma^2=\frac{1}{(1-\alpha)(\alpha\lambda^\alpha-\nu)},\ \mathrm{lb}=0,\ \mathrm{ub}=\pi\right).$$

Hence, according to the inequality (19), we have

$$\begin{aligned}
\frac{f(z,u)}{\hat{q}(z,u)} &= \frac{\mathrm{Erf}\left(\pi\sqrt{(1-\alpha)(\alpha\lambda^\alpha-\nu)/2}\right)\Gamma\left((1-\alpha)\lambda^\alpha-\frac{(1-\alpha)\nu}{\alpha}+1\right)}{G(\nu,\lambda)\sqrt{2\pi(1-\alpha)(\alpha\lambda^\alpha-\nu)}} \\
&\quad\times e^{\lambda^\alpha}B(u)^{\frac{\nu}{\alpha}}z^{-(1-\alpha)\lambda^\alpha}\exp\left(-\lambda B(u)^{\frac{1}{\alpha}}z^{-\frac{1-\alpha}{\alpha}}+\frac{(1-\alpha)(\alpha\lambda^\alpha-\nu)u^2}{2}\right) \\
&\le \frac{\mathrm{Erf}\left(\pi\sqrt{(1-\alpha)(\alpha\lambda^\alpha-\nu)/2}\right)\Gamma\left((1-\alpha)\lambda^\alpha-\frac{(1-\alpha)\nu}{\alpha}+1\right)}{G(\nu,\lambda)\Gamma((1-\alpha)\lambda^\alpha+1)\sqrt{2\pi(1-\alpha)(\alpha\lambda^\alpha-\nu)}} \\
&\quad\times(1-\alpha)^{\frac{(1-\alpha)\nu}{\alpha}}\alpha^\nu\Gamma((1-\alpha)\lambda^\alpha+1)(1-\alpha)^{-(1-\alpha)\lambda^\alpha}\lambda^{-\alpha(1-\alpha)\lambda^\alpha}e^{(1-\alpha)\lambda^\alpha} \\
&\le \frac{C_4(\alpha,\lambda)\Gamma\left((1-\alpha)\lambda^\alpha-\frac{(1-\alpha)\nu}{\alpha}+1\right)(1-\alpha)^{\frac{(1-\alpha)\nu}{\alpha}}\alpha^\nu}{G(\nu,\lambda)\Gamma((1-\alpha)\lambda^\alpha+1)\sqrt{1-\frac{\nu}{\alpha\lambda^\alpha}}} = \bar{C}_4,
\end{aligned}$$

where $\bar{C}_4$ is the associated A/R constant. In general, given parameters $\alpha, \lambda, \nu$, we choose the envelope with the smallest A/R constant to generate $G_{\alpha,\lambda,\nu}$. A combination of these four simulation schemes leads to a more efficient algorithm. □

In general, the additional parameter $\nu$ for the GTS distributions makes analyzing the complexity of Algorithm 4.1 more challenging, as the analytical form for $G(\nu,\lambda)$ is unknown. In the literature, the only existing algorithm for GTS distributions is the **decomposition scheme (DS)** proposed by Favaro et al. [2015]. The relevant numerical comparison tests between Algorithm 4.1 and the DS [Favaro et al. 2015] will be illustrated in Section 5.

## 5 NUMERICAL VERIFICATION AND COMPARISON

In this section, we provide numerical examples for sampling two families of tilted stable distributions: ETS and GTS distributions. The simulation experiments are all conducted on a normal laptop with the Intel Core i7-6500U CPU@2.50-GHz processor, 8.00 GB of RAM, Windows 10 Home, and a 64-bit operating system. The algorithms are coded and performed in R.3.4.2, and computing time is measured by the elapsed *CPU time* in seconds. Numerical validation and tests for the ETS algorithm are based on the **probability density function (PDF)**, **cumulative distribution function (CDF)**, and quantiles of $S_{\alpha,\lambda}$, which can be obtained by inverting Laplace transform (3) numerically. For the GTS simulation scheme, verifying via the CDF, PDF, and quantiles are non-executable, as its closed-form Laplace transform is not available. So we establish comparison tests for the empirical CDFs, PDFs, and quantiles generated by Algorithm 4.1 and by the DS of Favaro et al. [2015].

For Algorithm 3.1 of ETS distributions, the plots of CDFs and PDFs under parameter settings $\alpha = 0.3, 0.6$, $\lambda = 1.0, 5.0$ are provided in Figure 3. The Q-Q plots for the empirical quantiles of $S_{\alpha,\lambda}$ against the corresponding theoretical quantiles are presented in Figure 4, and the associated results in detail are reported in Table 1. We can see that our algorithm can achieve a very high level of accuracy, and the simulated CDFs, PDFs, and quantiles are fitted pretty well to the associated numerical inversions. There are a variety of available algorithms for numerically inverting Laplace transforms with high accuracy in the literature, such as Gaver [1966], Stehfest [1970], and Abate and Whitt [1992, 1995, 2006], to name a few. Here, we adopt the classical Euler scheme as described in Abate and Whitt [2006, Section 5, pp. 415-416].

To investigate the performance of our SR scheme for the ETS, we made a comparison of the CPU time for Algorithm 3.1 against the DR scheme for simulating 100,000 samples under the

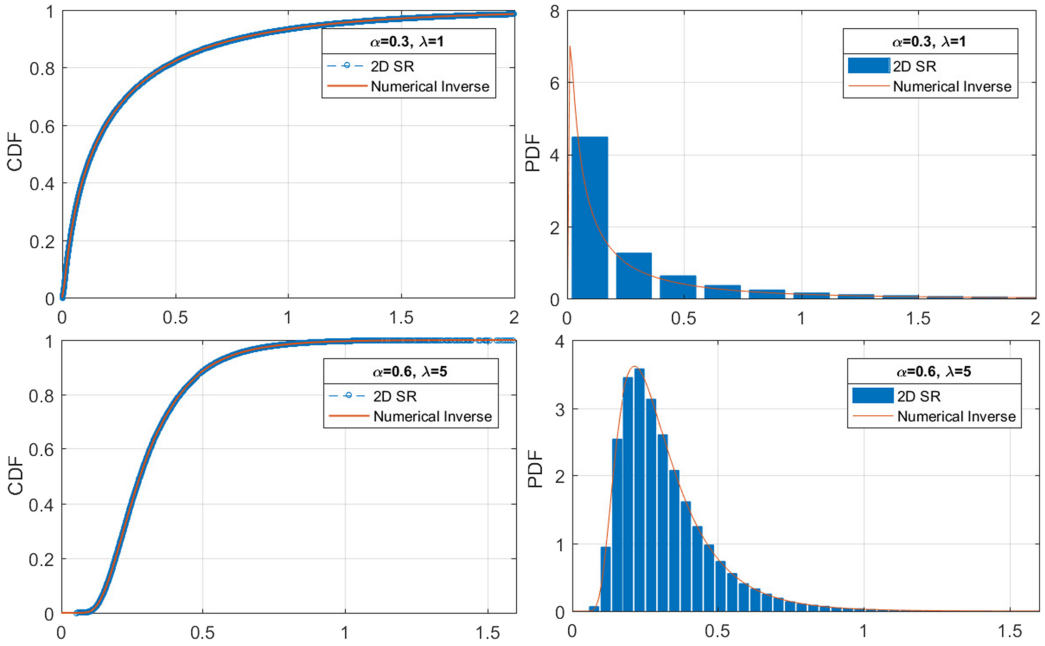


Fig. 3. Comparison of the empirical CDF/PDF for the SR scheme (via Algorithm 3.1) of $S_{\alpha,\lambda}$ with the CDF/PDF obtained via numerical inverse the Laplace transform of (3).

Table 1. Comparison of the Empirical Quantiles of $S_{\alpha,\lambda}$ for the SR Scheme (via Algorithm 3.1) Against the Theoretical Quantiles of $S_{\alpha,\lambda}$ Approximated via Numerical Inverse the Laplace Transform of (3)

| Quantile | 10% | 20% | 30% | 40% | 50% | 60% | 70% | 80% | 90% |
|---|---|---|---|---|---|---|---|---|---|
| | $\alpha = 0.3 \quad \lambda = 1$ | | | | | | | | |
| 2D SR | 0.0172 | 0.0342 | 0.0566 | 0.0877 | 0.1299 | 0.1912 | 0.2874 | 0.4451 | 0.7754 |
| Numerical Inverse | 0.0173 | 0.0342 | 0.0567 | 0.0877 | 0.1303 | 0.1913 | 0.2873 | 0.4452 | 0.7756 |
| | $\alpha = 0.6 \quad \lambda = 5$ | | | | | | | | |
| 2D SR | 0.1592 | 0.1905 | 0.2185 | 0.2466 | 0.2773 | 0.3125 | 0.3558 | 0.4163 | 0.5181 |
| Numerical Inverse | 0.1592 | 0.1905 | 0.2184 | 0.2466 | 0.2772 | 0.3125 | 0.3562 | 0.4163 | 0.5182 |

parameter settings $\alpha \in \{0.05, 0.1, \ldots, 0.9, 0.99\}$ and $\lambda \in \{0.01, 0.1, \ldots, 10^6\}$. The numerical results are reported in Table 2. We can see that our SR scheme performs well for all combinations of $\alpha$ and $\lambda$. The out-performance of our algorithm would even become much more substantial when $\alpha$ is close to 0. For example, it is about eight times faster than the DR scheme when $\alpha = 0.05$. In addition, Algorithm 3.1 is also very fast when the tilting parameter $\lambda$ is not very large, which clearly indicates that the acceptance rate of Algorithm 3.1 is higher than the DR scheme [Devroye 2009] for a small tilting parameter $\lambda$. Based on the DR scheme, Hofert [2011b] proposed a more efficient sampling algorithm for ETS distributions by combining the FR scheme with the DR scheme. Since the SR scheme outperforms the DR scheme over all combinations of parameters, the combination algorithms of Hofert [2011b] can be further improved by combining the SR scheme with the FR scheme.

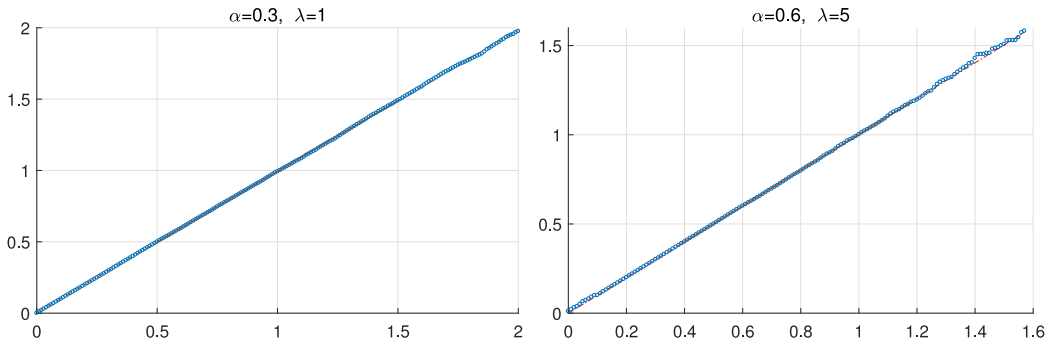


Fig. 4. Q-Q plots with the vertical axis being the empirical quantiles of $S_{\alpha,\lambda}$ for the SR scheme (via Algorithm 3.1) and the horizonal axis being the theoretical quantiles of $S_{\alpha,\lambda}$ approximated via numerical inverse the Laplace transform of (3).

Table 2. Comparison of CPU Time for Generating 100,000 Samples Based on the SR Scheme (via Algorithm 3.1) and the DR Scheme [Devroye 2009], Respectively

| $\alpha$ \ $\lambda$ | 0.01 | | 0.10 | | 1.00 | | 10 | | 100 | | 1,000 | | 10,000 | | 100,000 | | 1,000,000 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | SR | DR | SR | DR | SR | DR | SR | DR | SR | DR | SR | DR | SR | DR | SR | DR | SR | DR |
| 0.05 | 2.58 | 18.35 | 2.33 | 18.80 | 2.36 | 19.05 | 2.42 | 19.24 | 2.23 | 18.62 | 2.22 | 19.09 | 2.47 | 18.79 | 2.32 | 18.59 | 2.45 | 18.16 |
| 0.10 | 2.51 | 19.36 | 2.67 | 18.92 | 2.56 | 18.36 | 2.47 | 18.14 | 2.62 | 18.08 | 4.44 | 17.98 | 3.96 | 17.71 | 4.08 | 17.3 | 3.44 | 16.78 |
| 0.20 | 2.33 | 18.72 | 2.53 | 20.44 | 5.22 | 18.26 | 4.51 | 17.16 | 3.86 | 16.31 | 3.50 | 15.43 | 3.58 | 9.43 | 3.22 | 6.91 | 3.05 | 5.18 |
| 0.30 | 2.02 | 19.23 | 2.36 | 18.16 | 4.45 | 17.54 | 4.50 | 15.93 | 3.95 | 14.14 | 3.21 | 6.84 | 3.30 | 4.98 | 3.21 | 4.39 | 2.69 | 4.07 |
| 0.40 | 1.93 | 18.69 | 2.35 | 18.29 | 4.03 | 18.61 | 3.86 | 14.97 | 3.69 | 7.19 | 3.78 | 4.69 | 3.47 | 4.64 | 3.19 | 4.12 | 2.76 | 4.21 |
| 0.50 | 1.73 | 19.55 | 1.94 | 18.53 | 3.59 | 17.08 | 3.50 | 13.73 | 3.22 | 5.14 | 3.11 | 4.46 | 3.36 | 4.23 | 3.53 | 3.95 | 3.69 | 4.02 |
| 0.60 | 1.56 | 18.66 | 1.97 | 19.05 | 3.65 | 18.47 | 3.28 | 13.97 | 3.39 | 4.75 | 3.19 | 4.22 | 3.00 | 4.19 | 3.49 | 4.03 | 3.17 | 4.03 |
| 0.70 | 1.61 | 18.50 | 1.76 | 18.81 | 3.46 | 17.88 | 3.17 | 9.28 | 3.01 | 4.50 | 3.11 | 4.32 | 3.19 | 4.23 | 3.34 | 3.97 | 3.25 | 4.08 |
| 0.80 | 1.84 | 18.53 | 1.83 | 18.49 | 3.45 | 18.42 | 2.94 | 9.33 | 2.92 | 4.52 | 2.38 | 4.47 | 3.17 | 4.81 | 3.31 | 3.92 | 3.07 | 4.24 |
| 0.90 | 1.78 | 18.45 | 1.59 | 18.96 | 1.70 | 18.62 | 2.90 | 14.73 | 2.76 | 4.46 | 2.39 | 4.55 | 2.84 | 4.78 | 2.94 | 4.00 | 2.86 | 4.97 |
| 0.99 | 1.50 | 17.81 | 1.54 | 18.00 | 1.62 | 18.86 | 1.88 | 18.44 | 3.14 | 13.94 | 2.28 | 4.41 | 2.64 | 4.69 | 3.02 | 4.06 | 2.83 | 4.21 |

The comparison of empirical CDFs and CDFs for Algorithm 4.1 and the DS under various combinations of $(\alpha, \lambda, \nu)$ are illustrated in Figure 5. We also present the comparison of empirical quantiles in Figure 6 and report the associated results in Table 3. We can see that these two algorithms are closely matched in terms of CDF, PDF, and quantiles. Note that Algorithm 4.1 also has one special feature—that is, it can be used to sample $G_{\alpha,\lambda,\nu}$ for a negative $\nu$ such that $\nu > -\alpha\lambda^{\alpha}$. Figure 7 demonstrates the distributional behavior of this special class of GTS via its PDFs.

Meanwhile, we have also compared the simulation time for Algorithm 4.1 against the DS over a large range of values of $\alpha, \lambda, \nu$, and explore how the efficiency depends on them. The related numerical results are listed in Table 4. We see that our scheme is more efficient for most parameter settings provided in Table 4, especially for large values of $\alpha$, $\lambda$. For example, Algorithm 4.1 is extraordinarily fast when $\alpha = 0.6$ and $\lambda = 1,000$. In general, our proposed algorithm is significantly more efficient for a large range of parameter combinations. The key reason is that our SR scheme is developed independently and generates the GTS random variable directly without using the DR or FR method. This leads to a more straightforward procedure for implementation. In fact, the DS can be improved by generating the ETS random variable using our Algorithm 3.1, which would then speed up the entire simulation for the GTS random variable.

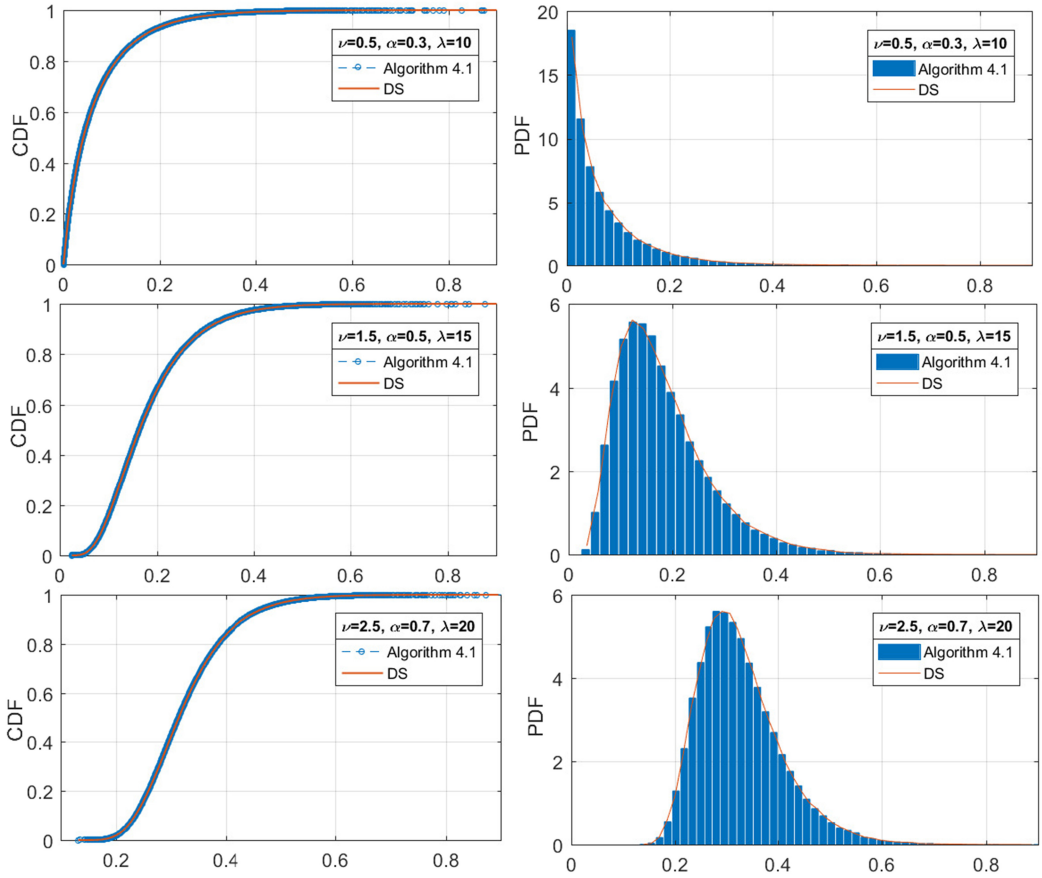


Fig. 5. Comparison of the empirical CDF/PDF for Algorithm 4.1 against the DS [Favaro et al. 2015].

Table 3. Comparison of the Empirical Quantiles of $G_{\alpha,\lambda,\nu}$ for Algorithm 4.1 Against the DS [Favaro et al. 2015]

| Quantile | 10% | 20% | 30% | 40% | 50% | 60% | 70% | 80% | 90% |
|---|---|---|---|---|---|---|---|---|---|
| | | | | $\nu = 0.5$ | $\alpha = 0.3$ | $\lambda = 10$ | | | |
| Algo. 4.1 | 0.0164 | 0.0271 | 0.0384 | 0.0510 | 0.0660 | 0.0840 | 0.1079 | 0.1420 | 0.2013 |
| DS | 0.0165 | 0.0272 | 0.0384 | 0.0509 | 0.0660 | 0.0844 | 0.1079 | 0.1419 | 0.2012 |
| | | | | $\nu = 1.5$ | $\alpha = 0.5$ | $\lambda = 15$ | | | |
| Algo. 4.1 | 0.0871 | 0.1081 | 0.1263 | 0.1441 | 0.1632 | 0.1849 | 0.2107 | 0.2451 | 0.3001 |
| DS | 0.0871 | 0.1083 | 0.1263 | 0.1443 | 0.1631 | 0.1849 | 0.2107 | 0.2450 | 0.3001 |
| | | | | $\nu = 2.5$ | $\alpha = 0.7$ | $\lambda = 20$ | | | |
| Algo. 4.1 | 0.2353 | 0.2583 | 0.2776 | 0.2955 | 0.3138 | 0.3336 | 0.3566 | 0.3861 | 0.4315 |
| DS | 0.2352 | 0.2583 | 0.2779 | 0.2954 | 0.3140 | 0.3334 | 0.3569 | 0.3862 | 0.4312 |

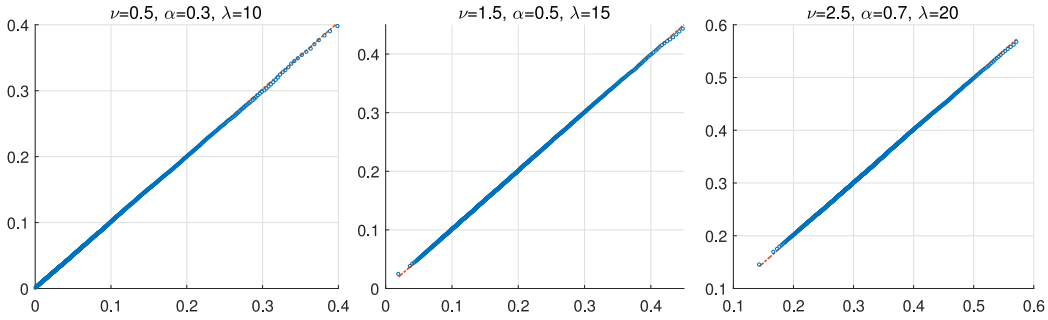


Fig. 6. Q-Q plots with the vertical axis being the empirical quantiles of $G_{\alpha,\lambda,\nu}$ for Algorithm 4.1 and the horizonal axis being the empirical quantiles of $G_{\alpha,\lambda,\nu}$ for DS [Favaro et al. 2015].

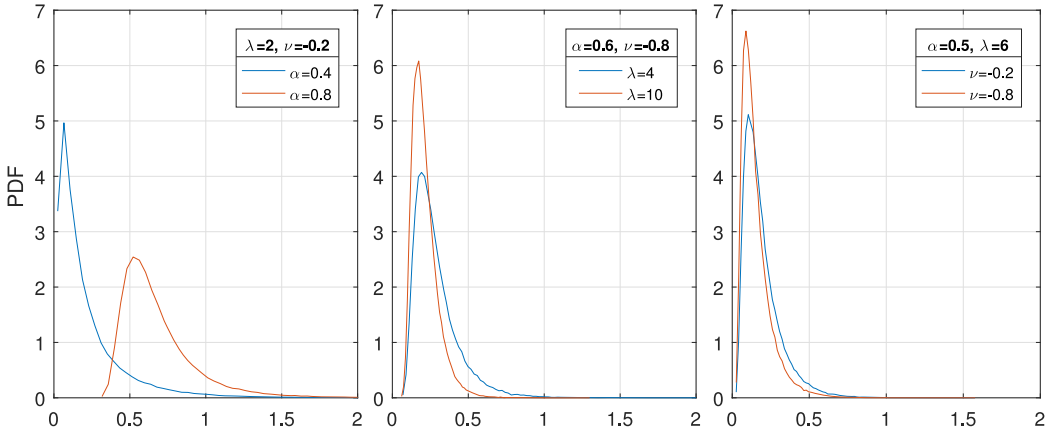


Fig. 7. PDFs of GTS with $\nu < 0$.

Table 4. Comparison of CPU Time for Generating 100,000 Samples Based on Algorithm 4.1 and the DS [Favaro et al. 2015], Respectively

| | $\nu = 1$ | | $\nu = 1.5$ | | $\nu = 2$ | | $\nu = 2.5$ | | $\nu = 3$ | | $\nu = 3.5$ | | $\nu = 4$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda$ | Algo 4.1 | DS | Algo 4.1 | DS | Algo 4.1 | DS | Algo 4.1 | DS | Algo 4.1 | DS | Algo 4.1 | DS | Algo 4.1 | DS |
| $\alpha = 0.2$ | | | | | | | | | | | | | | |
| 10 | 4.35 | 6.89 | 4.56 | 8.31 | 4.81 | 12.57 | 5.45 | 12.44 | 5.67 | 16.17 | 5.75 | 16.71 | 5.95 | 20.77 |
| 100 | 3.68 | 7.48 | 3.76 | 7.81 | 4.15 | 12.19 | 4.56 | 12.84 | 4.54 | 17.58 | 4.82 | 16.80 | 4.75 | 21.02 |
| 1,000 | 3.25 | 7.22 | 3.35 | 8.07 | 3.43 | 12.33 | 3.45 | 13.05 | 3.92 | 16.76 | 4.23 | 18.12 | 4.35 | 21.72 |
| $\alpha = 0.4$ | | | | | | | | | | | | | | |
| 10 | 4.43 | 7.02 | 4.67 | 7.98 | 5.56 | 12.97 | 6.43 | 13.95 | 6.89 | 17.14 | 7.80 | 17.70 | 8.64 | 21.11 |
| 100 | 3.35 | 6.88 | 3.56 | 7.79 | 3.65 | 12.89 | 4.22 | 13.08 | 4.77 | 16.97 | 5.23 | 16.64 | 5.54 | 21.01 |
| 1,000 | 3.28 | 6.81 | 3.49 | 8.88 | 3.34 | 11.89 | 3.54 | 13.88 | 3.53 | 16.31 | 3.89 | 18.14 | 4.23 | 21.00 |
| $\alpha = 0.6$ | | | | | | | | | | | | | | |
| 10 | 5.53 | 7.27 | 5.42 | 7.89 | 6.32 | 12.89 | 7.35 | 13.00 | 8.23 | 16.89 | 10.08 | 17.16 | 11.82 | 21.49 |
| 100 | 3.23 | 6.72 | 4.23 | 12.28 | 4.13 | 12.08 | 4.23 | 14.22 | 4.13 | 16.44 | 4.56 | 17.64 | 5.23 | 20.89 |
| 1,000 | 3.44 | 7.73 | 3.45 | 1540.32 | 3.25 | 12.16 | 3.62 | 186.15 | 3.15 | 16.52 | 3.86 | 54.73 | 3.65 | 20.67 |

## 6 CONCLUSION

In this article, a new efficient simulation scheme has been developed for sampling ETS and GTS distributions. The two important distributions appear routinely in financial applications and other areas that heavily rely on Monte Carlo simulation. The key principle of this approach is two-dimensional SR, which is very different from other existing schemes in the literature. The complexity of our new algorithm for the ETS family is uniformly bounded over all ranges of parameters. Remarkably, it beats all other algorithms. Our further extension for exactly sampling of the GTS family does not rely on sampling the ETS family, and hence our algorithm for the GTS family is more efficient than the DS (which is the only alternative algorithm in the literature). For future research, our algorithms can be adopted for further generating ETS-driven or GTS-driven stochastic processes as mentioned early in Section 1, which could lead many simulation-based applications.

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